Rank- Nullity Theorem: Unit I

Nullity of T and Rank of T.

Let V and W be vector spaces and
$T: V \to W$ be a Linear Transformation.
The dimension of $N(T)$ is called Nullity of T.
The dimension of $R(T)$ is called Range of T.

$N(T)$ is a set which contains non zero vectors. The number of these Non-zero vectors is the dimension of $N(T)$ and it is the Nullity of T.

No of elements in the basis of $R(T)$ is the dimension of $R(T)$.

Dimension Theorem:

Let V and W be vector spaces and $T: V \to W$ be a Linear Transformation.
If V is finite dimensional vector space, then

$$\boxed{\dim V = \text{Rank of } T + \text{Nullity of } T.}$$

F

Example:

Let $T: R^2 \to R^3$ defined by

$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

Verify dimension theorem.

<u>Solution:</u> R^2 and R^3 are vector spaces.

T is Linear Transformation.

$\dim(R^2) = 2 \Rightarrow \dim V = 2$

Let basis be $\beta = \{ \cancel{(1,0)}\ (1,0), (0,1) \}$

$R(T) = \text{span}(T(\beta))$

$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

$T(1,0) = (1+0, 0, 2(1) - 0)$

$= (1, 0, 2)$

$T(0,1) = (0+1, 0, 2(0) - 1)$

$= (1, 0, -1)$

$T(\beta) = \{ (1,0,2), (1,0,-1) \}$

$a(1,0,2) + b(1,0,-1) = (0,0,0)$

$(a, 0, 2a) + (b, 0, -b) = (0,0,0)$

$(a+b, 0+0, 2a-b) = (0,0,0)$

$\cancel{a+b}\ (a+b, 0, 2a-b) = (0,0,0)$

$\Rightarrow a+b = 0$

$2a - b = 0$

Solving $a+b=0$

$2a-b=0$

$$\boxed{a=0,\quad b=0}$$

$\therefore$ $(1,0,2)$ and $(1,0,-1)$ ~~are~~ are Linearly independent.

$\therefore$ $\{(1,0,2),(1,0,-1)\}$ is a basis for $R(T)$

Also $\{(1,0,2),(1,0,-1)\}$ generate $R(T)$

$\therefore \dim R(T) = 2$

Since $\dim R(T) = $ Rank of T

$\therefore$ $\boxed{\text{Rank of } T = 2.}$

~~By Rank - Nullity Theorem~~

dim V = ~~dim R(T) + Rank of T + Nullity of T.~~

To find ~~2~~ Nullity of T:

Definition:

$$N(T) = \{x \in V : T(x) = 0\}$$

$$N(T) = \{x = (a_1,a_2) \in \mathbb{R}^2 : T(a_1,a_2) = 0\}$$

$$= \{(a_1,a_2) \in \mathbb{R}^2 : (a_1+a_2,\ 0,\ 2a_1-a_2) = 0 = (0,0,0)\}$$

$$= \{(a_1,a_2) \in \mathbb{R}^2 ;\ a_1+a_2=0,\ 2a_1-a_2=0\}$$

$\Rightarrow a_1 = -a_2$, $\quad 2a_1-(-a_1)=0$

$3a_1 = 0$

$\Rightarrow a_1 = 0\}$

$\therefore N(T) = \{\overset{(a_1,a_2)}{(0,0)}\} = \{\bar{0}\}$

Scanned by CamScanner

$\dim N(T) =$ No of Non Zero Vectors in $N(T)$

$= 0 \Rightarrow$ Nullity of $T = 0$.

~~By Rank Nullit~~

$\therefore \dim V =$ Rank T + Nullity T.

$2 = 2 + 0$

$2 = 2$

$\therefore$ Rank-Nullity Theorem is verified.

If $T: R^3 \to R^2$ be a LT defined by.

$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$. Find Rank of T and Nullity of T.

Let $\alpha \in R^3 \Rightarrow \alpha = (x_1, x_2, x_3)$

Suppose $\alpha \in N(T)$

$\Rightarrow$ ~~N(T)~~ $T(\alpha) = \bar{0}$, $\bar{0} \in R^2$

$\Rightarrow T(x_1, x_2, x_3) = \bar{0}$

$\Rightarrow (x_1 - x_2, x_1 + x_3) = (0, 0)$

$\Rightarrow x_1 - x_2 = 0$ and $x_1 + x_3 = 0$

$\Rightarrow x_1 = x_2$ and $x_3 = -x_1$

If $x_1 = k$, $k \in R$

then $\alpha = (x_1, x_2, x_3)$

$= (x_1, x_1, -x_1)$

$= (k, k, -k)$, $k \in R$

$= k(1, 1, -1)$, $k \in R$

~~$\therefore$ If $\alpha \in N(T)$~~

$\therefore \alpha \in N(T) \Rightarrow \alpha = k(1, 1, -1), \; k \in R.$

$\therefore$ Every vector of $N(T)$ can be spanned by a single vector $\{(1, 1, -1)\}$

$\therefore \dim N(T) = 1 \Rightarrow$ Nullity of $T = 1.$

To find Rank of T:

By Rank Nullity Theorem

$\dim R^3 = \rho(T) + \text{Nullity of } T.$

$3 = \rho(T) + 1$

$\Rightarrow \boxed{\rho(T) = 2}$

Verify Rank-Nullity Theorem for L.T.

$T : R^3 \to R^3$ defined by

$T(x, y, z) = (x-y, 2y+z, x+y+z)$

Solution:

Given that $T : R^3 \to R^3$ by

$T(x, y, z) = (x-y, 2y+z, x+y+z)$

$\dim R^3 = 3.$

Suppose $\alpha \in N(T)$

$\Rightarrow T(\alpha) = \bar{0}$

$\Rightarrow T(x, y, z) = \bar{0}$

$\Rightarrow (x-y, 2y+z, x+y+z) = (0, 0, 0)$

$x-y=0, \quad 2y+z=0, \quad x+y+z=0$

$\Rightarrow x = y, \; -2y = z \;\Rightarrow y \quad z = -2y \Rightarrow$

$x = y$ and $z = -2y \Rightarrow z = -2x$

$\therefore \alpha = (x, y, z) = \alpha = (x, x, -2x)$

Suppose $x = k$, then $\alpha = (k, k, -2k)$

$= k(1, 1, -2)$

$\therefore \alpha \in N(T) \Rightarrow \alpha = k(1, 1, -2), \; k \in R$

$\therefore$ Every element of $N(T)$ is spanned by a single vector $(1, 1, -2)$

$\therefore$ Basis of $N(T) = \{(1, 1, -2)\}$

$\therefore \dim N(T) = 1$

$\Rightarrow$ Nullity of $T = 1$

~~By~~ ~~R~~ ~~di~~ $\dim R^3 =$ Rank of $T +$ Nullity T

$3 = \rho(T) + 1$

$\rho(T) = 3 - 1 = 2$

$\boxed{\rho(T) = 2}$

To find: $\rho(T)$

Standard basis of R^3

$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Trivially $S \subset R^3$ $\quad T(x, y, z) = (x - y, 2y + z, x + ...)$

~~T(x~~, $T(1, 0, 0) = (1 - 0, 0, 1) = (1, 0, 1)$

$T(0, 1, 0) = (-1, 2, 1)$

$T(0, 0, 1) = (0, 1, 1)$

$(1, 0, 0), (0, 1, 0), (0, 0, 1) \in R^3$

$T(1, 0, 0), T(0, 1, 0), T(0, 0, 1) \in$ ~~$R(T^3)$~~ ~~$R(T$~~ $R(T)$

$\therefore S' = \{(1,0,1), (-1,2,1), (0,\phi,1)\}$
$\subset R(T)$

By writing these vectors as rows of a matrix

$$A = \left\{\begin{array}{ccc} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right\} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Write in Row-Echelon Form

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & \phi \end{bmatrix} \quad \begin{array}{l} R_1' = R_1 \\ R_2' = R_2 + R_1 \\ R_3' = R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_1'' = R_1' \\ R_2'' = R_2' \\ R_3'' = 2R_3' - R_2' \end{array}$$

$\rho(T) =$ No of non zero rows in the echelon form

$\therefore \rho(T) = 2$

~~Rank~~. Also $\dim V = \dim(R^3) = 3$.

$\therefore$ By Rank Nullity Theorem

$\dim V =$ Rank $T +$ Nullity T.

$3 = 2 + 1$

$3 = 3$

Verified.